

Graded Polynomial Identities for Matrices with the Transpose Involution over an Infinite Field

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Abstract

Let F be an infinite field, and let $M_n(F)$ be the algebra of $n \times n$ matrices over F . Suppose that this algebra is equipped with an elementary grading whose neutral component coincides with the main diagonal. In this paper, we find a basis for the graded polynomial identities of $M_n(F)$ with the transpose involution. Our results generalize for infinite fields of arbitrary characteristic previous results in the literature which were obtained for the field of complex numbers and for a particular class of elementary G-gradings.

1 Introduction

Let F be a field and A be an F -algebra. A polynomial identity of the algebra A is a polynomial in noncommuting variables which vanishes under

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any substitution of these variables by elements of A .

One of the first important results about polynomial identities in algebras is the Amitsur-Levitzki Theorem which proves that the standard polynomial of degree $2n$ is a polynomial identity for $M_n(F)$. Specht [16] raised the question whether the T-ideal of all polynomial identities of a given algebra is finitely generated as a T-ideal. This problem was answered by Kemer [13] using the characteristic zero some decades latter. Although Kemer proved that there always exists a finite basis for the identities of a given algebra in characteristic zero, it is very difficult problem to find any such basis. The explicit polynomial identities for concrete algebras are known in very few cases. For instance, for $M_n(\mathbb{C})$ such a basis is known only if $n = 1$ or 2 . In light of this, mathematicians started to work with ‘weaker’ polynomial identities such as identities with trace, identities with involution and graded identities. It is worth mentioning that the graded identities were also used by Kemer in the solution of the Specht problem.

Subsequent to the pioneering work of Di-Vincenzo [6] about graded identities, many authors described the graded identities of $M_n(F)$ and other different important algebras in different contexts [1], [2], [3], [7], [17], [8], [4] and [18]. Also, identities with involution for $M_n(F)$ have been studied by some authors [12].

Recently Haile and Natapov [11] exhibited a basis for the graded identities with involution of $M_n(\mathbb{C})$, endowed with the transpose involution and with a crossed-product grading. This grading is an elementary grading of $M_n(F)$ by a group $G = \{g_1, \dots, g_n\}$ induced by the n -tuple (g_1, \dots, g_n) . The authors used the graph theory as in [10].

In this paper we generalize the results of Haile and Natapov for a broader class of gradings and for infinite fields of arbitrary characteristic. The main tool here is the use of generic matrices, and we use ideas similar to those in [1], [2], [4], [7], and [8].

2 Preliminaries

We denote by F an infinite field of arbitrary characteristic. All vector spaces and algebras are over F . We denote the algebra of $n \times n$ matrices over F by $M_n(F)$ and a group with the unity e by G .

If A is an algebra and G is a group, a G -grading on A is a decomposition of A as a direct sum of subspaces $A = \bigoplus_{g \in G} A_g$, indexed by elements of the group G , which satisfy $A_g A_h \subseteq A_{gh}$, for any $g, h \in G$. If $a \in A_g - \{0\}$, for some $g \in G$, we say that a is homogeneous of degree g and we denote

$\deg(a) = g$. The support of the grading, is the subset of G , $\text{Supp}(A) = \{g \in G; A_g \neq \{0\}\}$.

If $1 \leq i, j \leq n$, we denote by e_{ij} the matrix with 1 on the position (i, j) , and 0 elsewhere. We call them *elementary matrices*, or *matrix units*.

Now let $(g_1, \dots, g_n) \in G^n$ be an n -tuple of elements of G . For each $g \in G$, let $R_g \subseteq M_n(F)$ be the subspace generated by the elementary matrices e_{ij} for i and j satisfying $g_i^{-1}g_j = g$. Then $M_n(F) = \bigoplus_{g \in G} R_g$ is a G -grading on $M_n(F)$ called *elementary grading defined by (g_1, \dots, g_n)* .

We recall a known result from [5], which characterizes elementary gradings on $M_n(F)$.

Theorem 2.1. *If G is any group, a G -grading of $M_n(F)$ is elementary if and only if all matrix units e_{ij} are homogeneous.*

An *involution* on an algebra A is an antiautomorphism of the order two, that is, a linear map $*$: $A \rightarrow A$ satisfying $(ab)^* = b^*a^*$ and $(a^*)^* = a$, for all $a, b \in A$. A classic example of involution on $M_n(F)$ is the transpose map.

A G -graded algebra $A = \bigoplus_{g \in G} A_g$ with involution $*$ is called a *degree-inverting involution algebra* if $(A_g)^* = A_{g^{-1}}$ for all $g \in G$. In this case, we say that $*$ is a degree-inverting involution on A . In this paper, if A is a degree-inverting involution algebra, we say it is a $(G, *)$ -algebra. A typical example of a $(G, *)$ -algebra is $M_n(F)$ endowed with an elementary grading and with the transpose involution. The degree-inverting involutions on $M_n(F)$ have been described by the authors in [9].

Remark 2.2. *When dealing with identities with involution on algebras over fields of characteristic different from 2, one usually consider the decomposition $A = A^+ \oplus A^-$, where $A^+ = \{a \in A \mid a^* = a\}$ (symmetric component) and $A^- = \{a \in A \mid a^* = -a\}$ (skew-symmetric component) and set in the free algebra, the set of symmetric and skew-symmetric variables. Note that one cannot use this approach in the present case, since the symmetric and skew-symmetric components are no longer homogeneous. In order to deal with our case, we need to consider a free algebra where the grading and the involution behave in the same way as in the algebra we want to study its identities.*

2.1 The free $(G, *)$ -algebra and $(G, *)$ -identities

To describe the identities of $M_n(F)$ as a $(G, *)$ -algebra, we define what we call the *free $(G, *)$ -algebra*.

For each $g \in G$, we define two countable sets $X_g = \{x_{k,g}; k \in \mathbb{N}\}$ and $X_g^* = \{x_{k,g}^*; k \in \mathbb{N}\}$. Then, let $X = \bigcup_{g \in G} X_g$ and $X^* = \bigcup_{g \in G} X_g^*$.

Consider the free associative algebra $F\langle X \cup X^* \rangle$, which is freely generated by $X \cup X^*$. Of course, it is an algebra with an involution defined in a natural way. Now, we define a G -grading on this free algebra to make it a $(G, *)$ -algebra. Let $\deg(1) = e$, and for each $k \in \mathbb{N}$ and $g \in G$, let $\deg(x_{k,g}) = g$ and $\deg(x_{k,g}^*) = g^{-1}$. If $m = x_{i_1, g_1}^{\varepsilon_1} \cdots x_{i_l, g_l}^{\varepsilon_l}$ is a monomial in $F\langle X \cup X^* \rangle$, where ε_i is $*$ or nothing, we define $\deg(m) = \deg(x_{i_1, g_1}^{\varepsilon_1}) \cdots \deg(x_{i_l, g_l}^{\varepsilon_l})$.

If we define

$$(F\langle X \cup X^* \rangle)_g = \text{span}_F \{m = x_{i_1, g_1}^{\varepsilon_1} \cdots x_{i_l, g_l}^{\varepsilon_l} \mid \deg(m) = g\},$$

we obtain that $F\langle X \cup X^* \rangle = \bigoplus_{g \in G} (F\langle X \cup X^* \rangle)_g$ is a G -grading on the algebra $F\langle X \cup X^* \rangle$, which makes it a $(G, *)$ -algebra. We denote such algebra by $F\langle X \mid (G, *) \rangle$ and call it the free $(G, *)$ -algebra. The elements of $F\langle X \mid (G, *) \rangle$ are called $(G, *)$ -polynomials.

If A and B are G -graded algebras with involution, we say that a homomorphism $\phi : A \rightarrow B$ is a homomorphism of graded algebras with involutions, if $\phi(A_g) \subset B_g$, for all $g \in G$ and $\phi(x^*) = \phi(x)^*$, for all $x \in A$.

The algebra $F\langle X \mid (G, *) \rangle$ satisfies a universal property: for any $(G, *)$ -algebra A and for any map $\varphi : X \rightarrow A$ such that for all $g \in G$, $\varphi(X_g) \subseteq A_g$, there exists a unique homomorphism of graded algebras with involution $\phi : F\langle X \mid (G, *) \rangle \rightarrow A$, such that for all $x \in X$, $\phi(x) = \varphi(x)$.

Let A be $(G, *)$ -algebra. A polynomial $f \in F\langle X \mid (G, *) \rangle$ is called a $(G, *)$ -polynomial identity of A if $f \in \text{Ker}(\phi)$ for any homomorphism of graded algebra with involution $\phi : F\langle X \mid (G, *) \rangle \rightarrow A$. Equivalently, f vanishes under any admissible substitution of variables by the elements of A with the condition that if $x_{k,g}$ is substituted by $a \in A_g$, then $x_{k,g}^*$ is substituted by a^* .

We observe that if A is a $(G, *)$ -algebra, then it is a graded algebra, and if f is a graded polynomial identity of A , then it is also a $(G, *)$ -identity of A . In particular Proposition 4.1 of [7] also holds for $(G, *)$ -algebras.

Proposition 2.3. *Let G be a group and let $\bar{g} = (g_1, \dots, g_n) \in G^n$ be an n -tuple of elements from G . Suppose $M_n(F)$ is endowed with elementary grading induced by \bar{g} . The following assertions are equivalent*

1. *The neutral component of $M_n(F)$ coincides with the main diagonal.*
2. *$x_{1,e}x_{2,e} - x_{2,e}x_{1,e}$ is a graded identity of $M_n(F)$.*
3. *The elements of \bar{g} are pairwise distinct.*

A (two-sided) ideal $I \subset F\langle X \mid (G, *) \rangle$ is called a T_G^* -ideal if I is closed under all $(G, *)$ -endomorphism of $F\langle X \mid (G, *) \rangle$. We denote the set of all

$(G, *)$ -identities of A by $T_G^*(A)$. Let $S \subset F\langle X | (G, *) \rangle$. We denote the intersection of all T_G^* -ideals containing S by $\langle S \rangle_{T_G^*}$. Notice that $T_G^*(A)$ and $\langle S \rangle_{T_G^*}$ are T_G^* -ideals of $F\langle X | (G, *) \rangle$. We say that $S \subset F\langle X | (G, *) \rangle$ is a basis for the $(G, *)$ -identities of A if $T_G^*(A) = \langle S \rangle_{T_G^*}$.

Proposition 2.4. *Let G be a group and let $M_n(F)$ be endowed with the elementary grading induced by an n -tuple (g_1, \dots, g_n) of pairwise distinct elements from G , and with the transpose involution. The following polynomials are $(G, *)$ -identities for $M_n(F)$*

$$x_{1,e}x_{2,e} - x_{2,e}x_{1,e} \quad (1)$$

$$x_{1,e} - x_{1,e}^* \quad (2)$$

$$x_{1,g}, g \notin \text{Supp}(M_n(F)) \quad (3)$$

$$x_{1,g}x_{2,g^{-1}}x_{3,g} - x_{3,g}x_{2,g^{-1}}x_{1,g}, g \neq e \quad (4)$$

For more details about identities 1 and 4, see [3, Lemma 4.1]. Identity 2 follows from Proposition 2.3. Also, [11, Remark 2 of Theorem 8] shows that identity 4 follows from identity 2.

When dealing with ordinary polynomials, it is well known that each T-ideal is generated by its multi-homogeneous polynomials. In the case of $(G, *)$ -polynomials, we need to slight modify this concept.

Definition 2.5. *Let $f = f(x_{1,g_1}, \dots, x_{n,g_n}, x_{1,g_1}^*, \dots, x_{n,g_n}^*) \in F\langle X | (G, *) \rangle$. Write f as*

$$f = \sum_{l=1}^k \lambda_l m_l$$

where $\lambda_l \in F - \{0\}$ and m_l are monomials in $F\langle X | (G, *) \rangle$. The polynomial f is called *strongly multi-homogeneous* if for each $t \in \{1, \dots, m\}$, $\deg_{x_{t,g_t}} m_i + \deg_{x_{t,g_t}^*} m_i = \deg_{x_{t,g_t}} m_j + \deg_{x_{t,g_t}^*} m_j$ for all $i, j \in \{1, \dots, k\}$. Here, the symbol $\deg_x m_i$ denotes the number of times the variable x appears in the monomial m_i .

Following the classic Vandermonde argument, we can prove that if I is a $(G, *)$ -ideal, then I is generated by its strongly multi-homogeneous polynomials.

3 The $*$ -graded identities of $M_n(F)$

We start this section with the following theorem of [11], which we aim to generalize for infinite fields and for a broader class of gradings by adding the

identities $x_g = 0$ for $g \notin \text{Supp}(M_n(F))$. We observe that in [11] the authors used the graph theory to prove this result.

Theorem 3.1 (Haile-Natapov, Theorem 8, [11]). *Let $G = \{g_1, \dots, g_n\}$ be a group of order n . The ideal of $(G, *)$ -identities of $M_n(\mathbb{C})$ endowed with the elementary grading induced by (g_1, \dots, g_n) and with the transpose involution is generated as a T_G^* -ideal by the following elements*

1. $x_{i,e}x_{j,e} - x_{j,e}x_{i,e}$
2. $x_{i,e} - x_{i,e}^*$

From now on, we consider $M_n(F)$ endowed with elementary grading induced by the n -tuple $\bar{g} = (g_1, \dots, g_n) \in G^n$ of pairwise distinct elements of G , and we denote $G_0 = \text{Supp}(M_n(F))$.

Let $g \in G$. We define

$$D(g) = \{i \in \{1, \dots, n\} \mid g_i g \in \{g_1, \dots, g_n\}\}$$

and

$$\text{Im}(g) = \{j \in \{1, \dots, n\} \mid g_j g^{-1} \in \{g_1, \dots, g_n\}\}.$$

Notice that $|D(g)| = |\text{Im}(g)|$, $D(g^{-1}) = \text{Im}(g)$ and $D(g) = \emptyset$ if and only if $g \notin G_0$. In that case, if $i \in D(g)$, there exists a unique $j \in \{1, \dots, n\}$ such that $g_i g = g_j$. If we define $j = \hat{g}(i)$, we obtain a bijective map

$$\begin{array}{ccc} \hat{g} : D(g) & \longrightarrow & \text{Im}(g) \\ i & \longmapsto & \hat{g}(i) \end{array}$$

Observe that for each $i \in \{1, \dots, n\}$, we have $e_{i\hat{g}(i)} \in (M_n(F))_g$ and for each g in support of $M_n(F)$, $\widehat{g^{-1}} = (\hat{g})^{-1}$.

Lemma 3.2. *Let $g, h \in G$. If there exists $i \in D(g) \cap D(h)$ such that*

$$\hat{g}(i) = \hat{h}(i),$$

then $g = h$.

Proof. Let $i \in D(\hat{g}) \cap D(\hat{h})$. If $j = \hat{g}(i) = \hat{h}(i)$, then $g_i g = g_j = g_i h$. We can conclude that $g = h$. □

Let $\Omega = \{y_{i,\widehat{g}(i)}^k | g \in G, i \in D(g), k \in \mathbb{N}\}$ be a set of commuting variables and $F[\Omega]$ be the algebra of commuting polynomials in Ω . We denote the set of all matrices over $F[\Omega]$ by $M_n(\Omega)$. As in the case of matrices over F , if $\overline{g} = (g_1, \dots, g_n)$ is an n -tuple of elements of G , then $M_n(\Omega)$ is endowed with an elementary G -grading induced by \overline{g} .

Definition 3.3. For each $g \in G_0$ and $j \in \mathbb{N}$, the elements of $M_n(\Omega)$,

$$A_{j,g} = \sum_{i \in D(g)} y_{i,\widehat{g}(i)}^j e_{i\widehat{g}(i)}$$

and

$$A_{j,g}^* = \sum_{i \in D(g^{-1})} y_{\widehat{g^{-1}}(i),i}^j e_{i\widehat{g^{-1}}(i)}$$

are called generic $(G, *)$ -matrices. The subalgebra of $M_n(\Omega)$ generated by $\{A_{j,g}, A_{j,g}^* | g \in G_0, j \in \mathbb{N}\}$ is called the algebra of generic $(G, *)$ -matrices and we denote it by Gen .

Lemma 3.4. Let $g, h \in G_0$. If $y_{i,k}^j \in \Omega$ is an entry of the matrices $A_{j,g}$ and $A_{j,h}$, then $g = h$.

Proof. If $y_{i,k}^j$ is an entry of $A_{j,g}$ and of $A_{j,h}$ then $k = \widehat{g}(i)$ and $k = \widehat{h}(i)$. Now Lemma 3.2 implies that $g = h$. \square

Using classical arguments, we can prove the following proposition.

Proposition 3.5. The relatively free algebra $F\langle X | (G, *) \rangle / T_G^*(M_n(F))$ is isomorphic to Gen . Furthermore, $T_G^*(M_n(F)) = T_G^*(Gen)$.

We now define the following maps, which by an abuse of notation will be also denoted by $*$

$$\begin{aligned} * : G &\longrightarrow G \\ g &\longmapsto g^* = g^{-1} \end{aligned}$$

$$\begin{aligned} * : \Omega &\longrightarrow \Omega \\ y_{k\widehat{g}(k)} &\longmapsto y_{k\widehat{g}(k)}^* = y_{\widehat{g^{-1}}(k)k} \end{aligned}$$

Given $h_1^{\varepsilon_1}, h_2^{\varepsilon_2}, \dots, h_r^{\varepsilon_r} \in G_0$, where $h_i \in G$ and ε_i is $*$ or nothing, we consider the composition $\nu = \widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}$ of the corresponding functions. This may not be well defined, and we will prove in Lemma 3.7 that in this case the monomial $x_{1,h_1}^{\varepsilon_1} \dots x_{r,h_r}^{\varepsilon_r}$ is a graded identity for $M_n(F)$. Otherwise, its domain $D_\nu = D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}}$ is the set of $i \in \{1, \dots, n\}$ for which the image $\widehat{h_r^{\varepsilon_r}}(\dots(\widehat{h_1^{\varepsilon_1}}(i))\dots)$ is well defined.

Lemma 3.6. *Let $g, h \in G$, then $D(\widehat{hg}) \subseteq D(\widehat{gh})$. Moreover, if $i \in D(\widehat{hg})$, then $\widehat{hg}(i) = \widehat{gh}(i)$.*

Proof. If $D(\widehat{hg}) = \emptyset$, the result is obvious. Suppose $D(\widehat{hg}) \neq \emptyset$. If $i \in D(\widehat{hg})$, let $k = \widehat{g}(i)$ and $j = \widehat{h}(k)$. Then, $g_k = g_i g$ and $g_j = g_k h$, and we obtain $g_j = g_i(gh)$, that is, $\widehat{gh}(i) = j$. \square

Lemma 3.7. *Let $h_1^{\varepsilon_1}, h_2^{\varepsilon_2}, \dots, h_r^{\varepsilon_r} \in G_0$. If $D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}} = \emptyset$ then*

$$A_{i_1, h_1}^{\varepsilon_1} A_{i_2, h_2}^{\varepsilon_2} \dots A_{i_r, h_r}^{\varepsilon_r} = 0.$$

Moreover, if the set $D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}}$ is nonempty then the i -th line of the matrix $A_{i_1, h_1}^{\varepsilon_1} A_{i_2, h_2}^{\varepsilon_2} \dots A_{i_r, h_r}^{\varepsilon_r}$ is nonzero if and only if $i \in D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}}$. In this case, if $j = \widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}(i)$, the only nonzero entry in the i -th line is a monomial of Ω in the j -th column.

Proof. The proof is by induction on the length r of the product. The result for $r = 1$ follows directly from Definition 3.3. Hence, we consider $r > 1$ and assume the result for products of length $r - 1$. Let us consider the first case $D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}} \neq \emptyset$. In this case $D_{\widehat{h_{r-1}^{\varepsilon_{r-1}}} \dots \widehat{h_1^{\varepsilon_1}}} \neq \emptyset$ and we denote $\nu = \widehat{h_{r-1}^{\varepsilon_{r-1}}} \dots \widehat{h_1^{\varepsilon_1}}$. The induction hypothesis implies that there exists monomials m_i , where $i \in D_{\widehat{h_{r-1}^{\varepsilon_{r-1}}} \dots \widehat{h_1^{\varepsilon_1}}}$, such that

$$A_{i_1, h_1}^{\varepsilon_1} A_{i_2, h_2}^{\varepsilon_2} \dots A_{i_r, h_r}^{\varepsilon_r} = \left(\sum_{i \in D_{\widehat{h_{r-1}^{\varepsilon_{r-1}}} \dots \widehat{h_1^{\varepsilon_1}}}} m_i e_{i\nu(i)} \right) \left(\sum_{j \in D_{\widehat{h_r^{\varepsilon_r}}}} (y_{j\widehat{h_r}(j)})^{\varepsilon_r} e_{j\widehat{h_r^{\varepsilon_r}}(j)} \right) \quad (5)$$

Note that $e_{i\nu(i)} e_{j\widehat{h_r^{\varepsilon_r}}(j)} \neq 0$ for some j if and only if $i \in D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}}$. In this case, the product equals $e_{i\widehat{h_r^{\varepsilon_r}}(j)}$. Hence, we obtain

$$A_{i_1, h_1}^{\varepsilon_1} A_{i_2, h_2}^{\varepsilon_2} \dots A_{i_r, h_r}^{\varepsilon_r} = \sum_{i \in D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}}} (m_i (y_{j\widehat{h_r}(j)})^{\varepsilon_r}) e_{i\widehat{h_r^{\varepsilon_r}}(\nu(i))},$$

and the result follows. Now, assume that $D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}} = \emptyset$. If $D_{\widehat{h_{r-1}^{\varepsilon_{r-1}}} \dots \widehat{h_1^{\varepsilon_1}}} = \emptyset$ then by the induction hypothesis $A_{i_1, h_1}^{\varepsilon_1} A_{i_2, h_2}^{\varepsilon_2} \dots A_{i_{r-1}, h_{r-1}}^{\varepsilon_{r-1}} = 0$ and the

result holds. Moreover, if $D_{\widehat{h_{r-1}^{\varepsilon_{r-1}} \dots h_1^{\varepsilon_1}}} \neq \emptyset$ then we may write the product $A_{i_1, h_1}^{\varepsilon_1} A_{i_2, h_2}^{\varepsilon_2} \dots A_{i_r, h_r}^{\varepsilon_r}$ as in (5). Since $D_{\widehat{h_r^{\varepsilon_r} \dots h_1^{\varepsilon_1}}} = \emptyset$, every product $e_{i\nu(i)} e_{j\widehat{h_r^{\varepsilon_r}(j)}}$ equals zero and therefore $A_{i_1, h_1}^{\varepsilon_1} A_{i_2, h_2}^{\varepsilon_2} \dots A_{i_r, h_r}^{\varepsilon_r} = 0$. \square

Definition 3.8. Suppose $\mathbf{h} = (h_1^{\varepsilon_1}, \dots, h_m^{\varepsilon_m}) \in G^m$ such that $D_{\widehat{h_m^{\varepsilon_m} \dots h_1^{\varepsilon_1}}} \neq \emptyset$.

For each $k \in D_{\widehat{h_m^{\varepsilon_m} \dots h_1^{\varepsilon_1}}}$, we denote by $s_k(\mathbf{h}) = (s_1^k, \dots, s_m^k, s_{m+1}^k)$ the following sequence, inductively by setting:

- (1) $s_1^k = k$
- (2) $s_r^k = \widehat{h_{r-1}^{\varepsilon_{r-1}}}(s_{r-1}^k)$, for $r \in \{2, \dots, m+1\}$.

We denote by $t_k(\mathbf{h}) = (t_1^k, \dots, t_m^k)$ the sequence defined by

$$t_r^k = \widehat{h_r}(s_r^k), \quad r \in \{1, \dots, m\}$$

Lemma 3.9. Let $h_1^{\varepsilon_1}, \dots, h_m^{\varepsilon_m} \in G$ such that $D_{\widehat{h_m^{\varepsilon_m} \dots h_1^{\varepsilon_1}}} \neq \emptyset$. Then

$$A_{1, h_1}^{\varepsilon_1} \dots A_{m, h_m}^{\varepsilon_m} = \sum_{k \in D_{\widehat{h_m^{\varepsilon_m} \dots h_1^{\varepsilon_1}}}} \omega_k^m e_{k, s_{m+1}^k},$$

where $\omega_k^m = (y_{s_1^k, t_1^k}^1)^{\varepsilon_1} \dots (y_{s_m^k, t_m^k}^m)^{\varepsilon_m}$. Furthermore, each matrix in the product $(A_{1, h_1}^{\varepsilon_1}, \dots, A_{m, h_m}^{\varepsilon_m})$ contributes with exactly one factor of it in the product $(y_{s_1^k, t_1^k}^1)^{\varepsilon_1} \dots (y_{s_m^k, t_m^k}^m)^{\varepsilon_m}$. For each $p \in \{1, \dots, m\}$, $A_{p, h_p}^{\varepsilon_p}$ contributes with $(y_{s_p^k, t_p^k}^p)^{\varepsilon_p}$.

Proof. The proof follows by induction on m . If $m = 1$, the result is obvious.

Suppose $m > 1$. By the induction hypothesis, we obtain

$$\begin{aligned} A_{1, h_1}^{\varepsilon_1} \dots A_{m, h_m}^{\varepsilon_m} &= (A_{1, h_1}^{\varepsilon_1} \dots A_{m-1, h_{m-1}}^{\varepsilon_{m-1}}) A_{m, h_m}^{\varepsilon_m} \\ &= \sum_{k \in D_{\widehat{h_{m-1}^{\varepsilon_{m-1}} \dots h_1^{\varepsilon_1}}}} \omega_k^{m-1} e_{k, s_m^k} \sum_{i \in D(\widehat{h_m^{\varepsilon_m}})} (y_{i, \widehat{h_m}(i)}^m)^{\varepsilon_m} e_{i \widehat{h_m}(i)} \\ &= \sum_{k \in D_{\widehat{h_{m-1}^{\varepsilon_{m-1}} \dots h_1^{\varepsilon_1}}}} \omega_k^{m-1} (y_{s_m^k, \widehat{h_m}(s_m^k)}^m)^{\varepsilon_m} e_{k, s_{m+1}^k} \end{aligned}$$

Now, the proof follows once one observes that $t_m^k = \widehat{h_m}(s_m^k)$ and $\omega_k^m = \omega_k^{m-1} (y_{s_m^k, t_m^k}^m)^{\varepsilon_m}$. \square

Definition 3.10. Let $\sigma \in S_m$. For $m = x_{i_{\sigma(1)}, h_{\sigma(1)}}^{\varepsilon_{\sigma(1)}} \cdots x_{i_{\sigma(n)}, h_{\sigma(n)}}^{\varepsilon_{\sigma(n)}}$, and any two integers $1 \leq k \leq l \leq n$, we denote $m^{[k, l]}$ the subword obtained from m by deleting the first $k - 1$ and the last $m - l$ variables.

$$m^{[k, l]} = x_{i_{\sigma(k)}, h_{\sigma(k)}}^{\varepsilon_{\sigma(k)}} \cdots x_{i_{\sigma(l)}, h_{\sigma(l)}}^{\varepsilon_{\sigma(l)}}.$$

Lemma 3.11. Let $x_{i_1, h_1}^{\varepsilon_1} \cdots x_{i_r, h_r}^{\varepsilon_r}$ and $x_{j_1, h'_1}^{\eta_1} \cdots x_{j_l, h'_l}^{\eta_l}$ be two monomials, with ε_k and η_k being $*$ or nothing, such that the matrices $A_{i_1, h_1}^{\varepsilon_1} \cdots A_{i_r, h_r}^{\varepsilon_r}$ and $A_{j_1, h'_1}^{\eta_1} \cdots A_{j_s, h'_s}^{\eta_s}$ have in the same position, the same nonzero entry. Then, $r = l$ and there exists a permutation $\sigma \in S_r$ such that $j_q = i_{\sigma(q)}$ and $h'_q = h_{\sigma(q)}$ for all $q \in \{1, \dots, r\}$. In particular, $x_{i_1, h_1}^{\varepsilon_1} \cdots x_{i_r, h_r}^{\varepsilon_r} - x_{j_1, h'_1}^{\eta_1} \cdots x_{j_l, h'_l}^{\eta_l}$ is a strongly multi-homogeneous polynomial. If this entry is $(y_{s_1^k, t_1^k}^{i_1})^{\varepsilon_1} \cdots (y_{s_r^k, t_r^k}^{i_r})^{\varepsilon_r}$, then $(y_{s_{\sigma(q)}^k, t_{\sigma(q)}^k}^{i_{\sigma(q)}})^{\varepsilon_{\sigma(q)}} = (y_{s'_q, t'_q}^{j_q})^{\eta_q}$ for $q = 1, \dots, r$.

Proof. Let $A_{i_1, h_1}^{\varepsilon_1} \cdots A_{i_r, h_r}^{\varepsilon_r} = \sum_{k \in D_{\widehat{h_m^{\varepsilon_r}} \cdots h_1^{\varepsilon_1}}} \omega_k^r e_{k, s_{r+1}^k}$, and $A_{j_1, h'_1}^{\eta_1} \cdots A_{j_l, h'_l}^{\eta_l} = \sum_{k \in D_{\widehat{h'_l}^{\eta_l} \cdots h'_1}^{\eta_1}} \tilde{\omega}_k^l e_{k, s_{m+1}^k}$ as in Lemma 3.9. Let k be the row in which these two matrices have the same nonzero entry. Let $\mathbf{h} = (h_1, \dots, h_r)$ and $\mathbf{h}' = (h'_1, \dots, h'_l)$ and consider the sequences $s_k(\mathbf{h}) = (s_1^k, \dots, s_{r+1}^k)$, $s_k(\mathbf{h}') = (s_1^k, \dots, s_{r+1}^k)$, $t_k(\mathbf{h}) = (t_1^k, \dots, t_{r+1}^k)$, $t_k(\mathbf{h}') = (t_1^k, \dots, t_{r+1}^k)$ as in Definition 3.8. Then $\omega_k^r = \tilde{\omega}_k^l$, where $\omega_k^r = (y_{s_1^k, t_1^k}^{i_1})^{\varepsilon_1} \cdots (y_{s_r^k, t_r^k}^{i_r})^{\varepsilon_r}$ and $\tilde{\omega}_k^l = (y_{s_1^k, t_1^k}^{j_1})^{\eta_1} \cdots (y_{s_l^k, t_l^k}^{j_l})^{\eta_l}$. Of course, we have $r = l$, and for each $q \in \{1, \dots, r\}$, there exists $p \in \{1, \dots, r\}$ such that $(y_{s'_q, t'_q}^{j_q})^{\eta_q} = (y_{s_p^k, t_p^k}^{i_p})^{\varepsilon_p}$.

Let us now consider two cases:

Case 1: $\varepsilon_p = \eta_q$. Then $i_p = j_q$, $s_p^k = s'_q$ and $t_p^k = t'_q$. Since $t'_q = \widehat{h'_q}(s'^k_q)$ and $t_p^k = \widehat{h_p}(s_p^k)$, we obtain $\widehat{h_p}(s_p^k) = \widehat{h'_q}(s_p^k)$ and Lemma 3.4 implies that $h_p = h'_q$.

Case 2: $\varepsilon_p \neq \eta_q$. We suppose $\varepsilon_p = *$. Then, we have

$$(y_{s'_q, \widehat{h'_q}(s'^k_q)}^{j_q}) = (y_{s_p^k, \widehat{h_p}(s_p^k)}^{i_p})^* = y_{\widehat{h_p^{-1}}(s_p^k), s_p^k}^{i_p}.$$

By comparing the indexes, we obtain $i_p = j_q$, $s_p^k = \widehat{h'_q}(s'^k_q)$ and $s'^k_q = \widehat{h_p^{-1}}(s_p^k)$.

Hence $s'^k_q = \widehat{h_p^{-1}}(\widehat{h'_q}(s'^k_q))$ and by Lemmas 3.2 and 3.6, we obtain $h'_q h_p^{-1} = e$ and $h'_q = h_p$.

From the above, we conclude that $x_{i_1, h_1}^{\varepsilon_1} \cdots x_{i_r, h_r}^{\varepsilon_r} - x_{j_1, h'_1}^{\eta_1} \cdots x_{j_l, h'_l}^{\eta_l}$ is a strongly multi-homogeneous polynomial. \square

Remark 3.12. Suppose that the same entry is (k, l) . Notice that there exist matrix units $e_{a_1 b_1} \in M_n(F)_{\alpha(x_{j_1, h'_1})}, \dots, e_{a_r b_r} \in M_n(F)_{\alpha(x_{j_r, h'_r})}$ such that $(e_{a_1 b_1})^{\eta_1} \cdots (e_{a_r b_r})^{\eta_r} = e_{kl}$ and $(e_{a_{\sigma(p)} b_{\sigma(p)}})^{\varepsilon_{\sigma(p)}} = (e_{a_p b_p})^{\eta_p}$ for $p = 1, \dots, r$.

4 The main theorem

We denote by J the T_G^* -ideal generated by the polynomials

$$x_{1,e} x_{2,e} - x_{2,e} x_{1,e} \text{ and } x_{1,e} - x_{1,e}^*.$$

In next two lemmas, we follow the ideas of [1], [7], and [18].

Lemma 4.1. Let $m_1(x_{i_1, h_1}^{\varepsilon_1}, \dots, x_{i_r, h_r}^{\varepsilon_r}), m_2(x_{i_1, h_1}^{\eta_1}, \dots, x_{i_l, h_l}^{\eta_l})$ be two monomials that start with the same variable and let \overline{m}_1 and \overline{m}_2 be the monomials obtained from m_1 and m_2 by deleting the first variable.

If $m_1(A_{i_1, h_1}^{\varepsilon_1}, \dots, A_{i_r, h_r}^{\varepsilon_r})$ and $m_2(A_{i_1, h_1}^{\eta_1}, \dots, A_{i_l, h_l}^{\eta_l})$ have in the same position the same non-zero entry, then $\overline{m}_1(A_{i_1, h_1}^{\varepsilon_1}, \dots, A_{i_r, h_r}^{\varepsilon_r})$ and $\overline{m}_2(A_{i_1, h_1}^{\eta_1}, \dots, A_{i_l, h_l}^{\eta_l})$ have in the same position the same non-zero entry.

Proof. It follows from Lemma 3.9. \square

Lemma 4.2. Let $m_1 = x_{i_1, h_1}^{\varepsilon_1} \cdots x_{i_r, h_r}^{\varepsilon_r}$ and $m_2 = x_{j_1, h'_1}^{\eta_1} \cdots x_{j_l, h'_l}^{\eta_l}$ be two monomials such that

$$A_{i_1, h_1}^{\varepsilon_1} \cdots A_{i_r, h_r}^{\varepsilon_r} \text{ and } A_{j_1, h'_1}^{\eta_1} \cdots A_{j_l, h'_l}^{\eta_l}$$

have in the same position, the same non-zero entry. Then $m_1 \equiv m_2 \pmod{J}$.

Proof. We prove this lemma by induction on n .

Suppose $n = 1$. If $A_{i_1, h_1}^{\varepsilon_1}$ and $A_{j_1, h'_1}^{\eta_1}$ have in the position (p, q) the same nonzero entry, then by Lemma 3.11, $i_1 = j_1$, and $h'_1 = h_1$. If $\varepsilon_1 = \eta_1$, then $m_1 = m_2$ and they are equivalent modulo J . If $\varepsilon_1 \neq \eta_1$, by comparing the (p, q) entries, we have $y_{p, \widehat{h_1(p)}}^{i_1} = y_{h_1^{-1}(p), p}^{i_1}$. Then, $\widehat{h_1(p)} = p$ and by Lemma 3.2, we obtain $h = e$, the neutral element of G . Hence, they are equivalent modulo J .

In proving the inductive step, we will show that m_2 is congruent, modulo J to a monomial m_3 that starts with the same variable of m_1 . Therefore, m_1 and m_3 will have in the same position, the same non-zero entry. According

to Lemma 4.1, $\overline{m_1}$ and $\overline{m_3}$ have in the same position the same non-zero entry. Thus, by induction, $m_1 \equiv m_3 \pmod{J}$, and consequently, $m_1 \equiv m_3 \equiv m_2 \pmod{J}$.

According to Lemma 3.11, $m_1 - m_2$ is a strongly multi-homogeneous polynomial. Furthermore, there exists a permutation $\sigma \in S_l$ such that $h_{i_{\sigma(s)}} = h'_{i_s}$, $i_{\sigma(s)} = j_s$, for all $s \in \{1, \dots, l\}$.

From Remark of Lemma 3.11, there exist matrix units

$$e_{a_1 b_1} \in M_n(F)_{\alpha(x_{j_1, h'_1})}, \dots, e_{a_l b_l} \in M_n(F)_{\alpha(x_{j_l, h'_l})}$$

such that $(e_{a_1 b_1})^{\eta_1} \dots (e_{a_l b_l})^{\eta_l} = e_{pq}$. Furthermore,

$$(e_{a_{\sigma(u)} b_{\sigma(u)}})^{\varepsilon_{\sigma(u)}} = (e_{a_u b_u})^{\eta_u} \text{ for } u = 1, \dots, l.$$

Suppose that the same entry of assumption is $(y_{s_1^k, t_1^k}^{i_1})^{\varepsilon_1} \dots (y_{s_l^k, t_l^k}^{i_l})^{\varepsilon_l}$ and the same position is (p, q) . Assume $\sigma^{-1}(1) = 1$. Note that $(y_{s_1^k, t_1^k}^{i_1})^{\varepsilon_1} = (y_{s_1^k, t_1^k}^{j_1})^{\eta_1}$, $j_1 = i_1$, and $h_1 = h'_1$. The letters $(y_{s_1^k, t_1^k}^{i_1})^{\varepsilon_1} = (y_{s_1^k, t_1^k}^{j_1})^{\eta_1}$ will appear in the p -th row of $A_{i_1, h_1}^{\varepsilon_1}$ and $A_{j_1, h'_1}^{\eta_1}$. Therefore, $\varepsilon_1 = \eta_1$ or $\varepsilon_1 \neq \eta_1$ and $h_1 = h'_1 = e$. The analysis of first situation is immediate. In the second, we have $m_2 \equiv (x_{j_1, h'_1}^{\eta_1})^* m_2^{[2, l]}$ by identity $x_{1, e} - x_{1, e}^*$.

Now, suppose that $\sigma^{-1}(1) > 1$. To analyze the monomial m_2 , we denote the number $\sigma^{-1}(1)$ by k_2 . Let t be the least positive integer such that $\sigma^{-1}(t+1) < \sigma^{-1}(1) \leq \sigma^{-1}(t)$. We denote the number $\sigma^{-1}(t+1)$ by k_1 and the number $\sigma^{-1}(t)$ by k_3 . We divide the rest of proof into 4 cases.

Case 1. $\eta_1 = \text{nothing}$ and $\varepsilon_1 = *$. In this situation, $b_1 = p$. If $\varepsilon_{\sigma(1)} = *$, then $b_{\sigma(1)} = p$. If $\varepsilon_{\sigma(1)} = \text{nothing}$, then $a_{\sigma(1)} = p$. Note that $\alpha(m_2^{[1, k_2]}) = \alpha(e_{a_{\sigma(1)} b_{\sigma(1)}}^{\varepsilon_{\sigma(1)}} \dots e_{a_{\sigma(k_2)} b_{\sigma(k_2)}}^{\varepsilon_{\sigma(k_2)}}) = \alpha(e_{pp}) = e$. Therefore,

$$m_2 \equiv m_4 = x_{j_{k_2}, h'_{k_2}} (m_2^{[1, k_2-1]})^* (m_2^{[k_2+1, l]}) \pmod{J}.$$

The last equivalence follows from identity $x_{1, e} - x_{1, e}^*$. If $h'_{k_2} = h_1 = e$, the result follows from identity $x_{1, e} - x_{1, e}^*$. Observe that the variables $x_{i_1, h_1}^{\varepsilon_1}$ and $x_{i_1, h_1} = x_{j_{k_2}, h'_{k_2}}$ could contribute with the same letter of $F[\Omega]$ in entry (p, q) . If this occurs, then $h'_{k_2} = h_1 = e$. From $x_{1, e} - x_{1, e}^*$, we obtain the desired result. Otherwise, suppose that $h_1 \neq e$, $x_{i_1, h_1}^{\varepsilon_1}$ and $x_{j_{k_2}, h'_{k_2}}$ do not contribute with the same letter of $F[\Omega]$. Thus, there exists an integer $w, w \neq k_2, 1 < w \leq l$, such that $x_{j_w, h'_w}^{\eta_w}$ and $x_{i_1, h_1}^{\varepsilon_1}$ contribute with the same

letter of $F[\Omega]$ in entry (p, q) . Without loss of generality, suppose that $w < k_2$. Notice that $\varepsilon_1 \neq \eta_w$ and $\alpha(m_4^{[1, w+1]}) = e$. Hence

$$m_2 \equiv m_4 \equiv x_{i_1, h_1}^{\varepsilon_1} (m_4^{[1, w]})^* m_4^{[w+2, l]} \mod J.$$

The Case 1 is verified.

Case 2. $\varepsilon_1 = \text{nothing}$ and $\eta_1 = *$. It is analogous to case 2.

Case 3. $\eta_1 = \varepsilon_1 = *$. Now, $\alpha(m_2^{[1, k_2-1]}) = g_{b_1} g_{b_1}^{-1} = e$. Suppose that $\eta_t = \eta_{t+1} = \text{nothing}$ (the other three cases are analogous). We analyze eight subcases.

Case 3.1: $k_1 = 1$.

Subcase 3.1.1: $\varepsilon_t \neq \eta_t$ and $\varepsilon_{t+1} = \eta_{t+1}$. Here, $\alpha(m_2^{[1, k_2-1]}) = \alpha(m_2^{[k_2, k_3-1]}) = e$. Consequently, by identity $x_{1,e} x_{2,e} - x_{2,e} x_{1,e}$, we have $m_2 \equiv m_2^{[k_2, k_3-1]} m_2^{[1, k_2-1]} m_2^{[k_3, l]} \mod J$.

Subcase 3.1.2: $\varepsilon_t = \eta_t$ and $\varepsilon_{t+1} = \eta_{t+1}$. This subcase is similar to Subcase 3.1.1. Here $\alpha(m_2^{[1, k_2-1]}) = \alpha(m_2^{[k_2, k_3]}) = e$.

Subcase 3.1.3: $\varepsilon_t \neq \eta_t$ and $\varepsilon_{t+1} \neq \eta_{t+1}$. Here, $\alpha(m_2^{[1, 1]}) = \alpha(m_2^{[2, k_2-1]})^{-1} = \alpha(m_2^{[k_2, k_3-1]})$. Hence, by identity $x_{1,e} - x_{1,e}^*$, we have $m_2 \equiv m_2^{[k_2, k_3-1]} (m_2^{[1, 1]})^* (m_2^{[2, k_2-1]})^* m_2^{[k_3, l]} \mod J$.

Subcase 3.1.4: $\varepsilon_t = \eta_t$ and $\varepsilon_{t+1} \neq \eta_{t+1}$. This subcase is analogous to Subcase 3.1.3. Now, $\alpha(m_2^{[1, 1]}) = \alpha(m_2^{[2, k_2-1]})^{-1} = \alpha(m_2^{[k_2, k_3]})$.

Case 3.2: $k_1 > 1$.

Subcase 3.2.1: $\varepsilon_t = \eta_t$ and $\varepsilon_{t+1} = \eta_{t+1}$. Here, $\alpha(m_2^{[1, k_1-1]}) = \alpha(m_2^{[k_1, k_2-1]})^{-1} = \alpha(m_2^{[k_2, k_3]})$. Thus, by identity $x_{1,e} - x_{1,e}^*$, we have $m_2 \equiv m_2^{[k_2, k_3]} (m_2^{[1, k_1-1]})^* (m_2^{[k_1, k_2-1]})^* m_2^{[k_3+1, l]} \mod J$.

Subcase 3.2.2: $\varepsilon_t \neq \eta_t$ and $\varepsilon_{t+1} = \eta_{t+1}$. Here, $\alpha(m_2^{[1, k_1-1]}) = \alpha(m_2^{[k_1, k_2-1]})^{-1} = \alpha(m_2^{[k_2, k_3-1]})$. Thus, by identity $x_{1,e} - x_{1,e}^*$, we have $m_2 \equiv m_2^{[k_2, k_3-1]} (m_2^{[1, k_1-1]})^* (m_2^{[k_1, k_2-1]})^* m_2^{[k_3, l]} \mod J$.

Subcase 3.2.3: $\varepsilon_t = \eta_t$ and $\varepsilon_{t+1} \neq \eta_{t+1}$. In this subcase, $\alpha(m_2^{[1, k_1]}) = \alpha(m_2^{[k_1+1, k_2-1]})^{-1} = \alpha(m_2^{[k_2, k_3]})$. Thus, by identity $x_{1,e} - x_{1,e}^*$, we have $m_2 \equiv m_2^{[k_2, k_3]} (m_2^{[1, k_1]})^* (m_2^{[k_1+1, k_2-1]})^* m_2^{[k_3, l]} \mod J$.

Subcase 3.2.4: $\eta_t \neq \varepsilon_t$ and $\eta_{t+1} \neq \varepsilon_{t+1}$. Finally,
 $\alpha(m_2^{[1,k_1]}) = \alpha(m_2^{[k_1+1,k_2-1]})^{-1} = \alpha(m_2^{[k_2,k_3-1]})$. In this way, by identity
 $x_{1,e} - x_{1,e}^*$, we have $n \equiv m_2^{[k_2,k_3-1]}(m_2^{[1,k_1]})^*(m_2^{[k_1+1,k_2-1]})^*m_2^{[k_3,l]}$
 $\text{mod } J$.

Case 4. $\eta_1 = \varepsilon_1 = \text{nothing}$. It is similar to case 3. □

We now recall the result [8, Corollary 3.2], which is based on an idea of [4, Corollary 11] about graded monomial identities.

Lemma 4.3. *If a monomial $x_{i_1,h_1} \dots x_{i_p,h_p}$ in $F\langle X \rangle$ is a graded identity for $M_n(F)$, then it is a consequence of a monomial in $T_G(M_n(F))$ of length at most $2n - 1$.*

By Lemma 3.7, a monomial $x_{i_1,h_1}^{\varepsilon_1} \dots x_{i_r,h_r}^{\varepsilon_r}$ is a $(G, *)$ -identity for $M_n(F)$ if and only if $D_{\widehat{h_r^{\varepsilon_r}} \dots \widehat{h_1^{\varepsilon_1}}} = \emptyset$. In particular, we obtain the following lemma.

Lemma 4.4. *A monomial $x_{i_1,h_1}^{\varepsilon_1} \dots x_{i_r,h_r}^{\varepsilon_r}$ is a $(G, *)$ -identity for $M_n(F)$ if and only if $x_{i_1,h_1^{\varepsilon_1}} \dots x_{i_r,h_r^{\varepsilon_r}}$ is a G -graded identity for $M_n(F)$.*

The following proposition is a straightforward consequence of the above lemmas.

Proposition 4.5. *Let m be a monomial identity of $M_n(F)$. Then, m is a consequence of monomial identities of degree up to $2n - 1$.*

Remark 4.6. *We recall that in [4] and [8], the authors conjecture that the graded monomial identities of $M_n(F)$ follow from the graded identities of degree up to n . We observe that once this conjecture is true, the same holds for the $(G, *)$ -identities of $M_n(F)$.*

We now state the main theorem of this paper.

Theorem 4.7. *Let U be the T_G^* -ideal generated by identities (1), (2), (3), and by the $(G, *)$ -monomial identities of degree up to $2n - 1$ of $M_n(F)$. Then,*

$$T_G^*(M_n(F)) = U.$$

Proof. From Proposition 2.4, $U \subseteq T_G^*(M_n(F))$. Let us suppose $U \subsetneq T_G^*(M_n(F))$. Then, there exists a strongly multi-homogeneous polynomial $f \in T_G^*(M_n(F)) - U$. By writing $f = \sum_{i=1}^l \lambda_i m_i$, we may suppose that

all $\lambda_i \in F - \{0\}$, $m_i \in F\langle X|(G, *) \rangle$ are monomials, which are not $(G, *)$ -identities for $M_n(F)$ and that the number l of nonzero summands of f is minimal among the strongly multi-homogeneous polynomials $f \in T_G^*(M_n(F)) - U$.

Since $f \in T_G^*(M_n(F))$, we have

$$\lambda_1 m_1 \equiv \sum_{i=2}^l -\lambda_i m_i \pmod{U}.$$

By substituting the variables by generic matrices, we obtain that m_1 and some m_j , $j > 1$, have the same nonzero entry in the same position. Hence, by Lemma 4.2, we conclude that $m_1 \equiv m_j \pmod{U}$.

Now let

$$h = (\lambda_j + \lambda_1)m_1 + \sum_{i \notin \{1, j\}} \lambda_i m_i$$

Then $h \equiv f \pmod{U}$ and the number of non-zero summands of h is $l - 1 < l$. This is a contradiction. \square

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